

# TETRAVALENT ARC-TRANSITIVE GRAPHS WITH UNBOUNDED VERTEX-STABILISERS

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**ABSTRACT.** It has long been known that there exist finite connected tetravalent arc-transitive graphs with arbitrarily large vertex-stabilisers. However, beside a well known family of exceptional graphs, related to the lexicographic product of a cycle with an edgeless graph on two vertices, only a few such infinite families of graphs are known. In this paper, we present two more families of tetravalent arc-transitive graphs with large vertex-stabilisers, each significant for its own reason.

## 1. INTRODUCTION

A celebrated theorem of Tutte [7, 8] states that, in a finite connected cubic arc-transitive graph, a vertex-stabiliser has order at most 48. As is well known and will be shown below, Tutte's result has no immediate generalization to graphs of valency 4. However, it is still an interesting question whether there exists a relatively tame function  $f$ , such that, for every connected tetravalent arc-transitive graph with  $n$  vertices, a vertex-stabiliser has order at most  $f(n)$ .

Here is a standard example showing that  $f$  must grow at least exponentially with the number of vertices: for  $r \geq 3$ , let  $W_r$  denote the lexicographic product  $C_r[\bar{K}_2]$  of a cycle of length  $r$  with an edgeless graph on two vertices. The graph  $W_r$  has vertex set  $\mathbb{Z}_r \times \mathbb{Z}_2$  with  $(v, i)$  adjacent to  $(v \pm 1, j)$  for  $v \in \mathbb{Z}_r$  and  $i, j \in \mathbb{Z}_2$ . This graph admits an arc-transitive action of the group  $G \cong C_2 \text{ wr } D_r$ , with the base group  $C_2^r \leq G$  preserving each fibre  $V_j = \{j\} \times \mathbb{Z}_2 \subseteq \mathbb{Z}_r \times \mathbb{Z}_2$  setwise and  $D_r$  acting naturally on the set of fibres  $\{V_j\}_j$ . The vertex-stabiliser  $G_v$  is then isomorphic to the group  $C_2^{r-1} \rtimes C_2$ . In particular, the order of  $G_v$  is  $2^r$  and grows exponentially with the number of vertices  $2r$  of the graph  $W_r$ . Further examples of families of tetravalent arc-transitive graphs exhibiting an exponential growth of  $|G_v|$  were found by Praeger and Xu in [5]. The graphs  $C(r, s)$ , constituting these families, will be described in Section 3.

On the other hand, it has been shown recently by the authors of this paper that a completely different picture emerges once the exceptional graphs  $C(r, s)$  are excluded [4, Corollary 3]. Namely, if  $\Gamma$  is a connected tetravalent  $G$ -arc-transitive graph not isomorphic to a graph  $C(r, s)$ , then either  $|G_v| \leq 2^4 3^6$  or

$$(1) \quad |\text{VT}| \geq 2|G_v| \log_2(|G_v|/2).$$

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Note that (1) implies that  $|G_v|$  is bounded above by a sub-linear function of  $|\text{V}\Gamma|$ . The first aim of this paper is to construct a family of connected  $G$ -arc-transitive graphs, not isomorphic to  $\text{C}(r, s)$ , attaining the bound given in (1). This will be done in Section 4 (see Definition 5). Both the graphs  $\text{C}(r, s)$  and the graphs presented in Section 4 have soluble groups of automorphisms.

The only family of connected tetravalent  $G$ -arc-transitive graphs with arbitrarily large vertex-stabilisers and with  $G$  non-soluble that was previously known to us is the family constructed by Conder and Walker in [2]. For a member  $\Gamma$  of this family, we have that  $G \cong \text{Sym}(n)$  (for some  $n$ ) and  $|\text{V}\Gamma| \geq (|G_v| - 1)!$ . In particular,  $|G_v|$  grows slower than any logarithmic function of  $|\text{V}\Gamma|$ .

The second family of graphs that we construct in this paper (see Definition 8) consists of tetravalent  $G$ -arc-transitive graphs  $\Delta_n$  with  $G \cong \text{Sym}(4n)$ . The vertex-stabiliser  $G_v$  in this family has order  $2^{2n}$ , and hence  $|\text{V}\Delta_n| = (4n)!/2^{2n}$ . Using Stirling's formula, one can see that, asymptotically,  $|G_v|$  grows slower than  $|\text{V}\Gamma|^c$  for any  $c > 0$ , but faster than any logarithmic function of  $|\text{V}\Gamma|$ . This is much slower than the growth from (1) but still considerably faster than the growth exhibited by the family in [2].

## 2. PRELIMINARIES

The following standard notation and terminology will be used throughout the paper. All the graphs will be finite, simple and connected. Let  $\Gamma$  be a graph and let  $G \leq \text{Aut}(\Gamma)$ . We say that  $\Gamma$  is  $G$ -arc-transitive provided that  $G$  acts transitively on the set of arcs of  $\Gamma$ . In this case, the permutation group  $G_v^{\Gamma(v)}$  induced by the action of the stabiliser  $G_v$  of a vertex  $v \in \text{V}\Gamma$  on the neighbourhood  $\Gamma(v)$  is transitive. A pair  $(\Gamma, G)$  is called *locally- $\text{D}_4$*  if  $\Gamma$  is a connected tetravalent  $G$ -arc-transitive graph with  $G_v^{\Gamma(v)}$  isomorphic to the dihedral group  $\text{D}_4$  in its action on 4 points. For a group  $G$ , a subgroup  $H$  and an element  $a \in G \setminus H$ , the *coset graph*  $\text{Cos}(G, H, a)$  is the graph with vertex set the set of right cosets  $G/H = \{Hg : g \in G\}$  and edge set  $\{\{Hg, Hag\} : g \in G\}$ . It was proved by Sabidussi [6] that every  $G$ -vertex-transitive graph is isomorphic to some coset graph of  $G$ . More precisely, we have the following well known result.

**Lemma 1.** *Let  $\Gamma$  be a connected  $G$ -arc-transitive graph, let  $G_v$  be the stabiliser of the vertex  $v \in \text{V}\Gamma$  and let  $a \in G$  be an automorphism with  $v^a \in \Gamma(v)$ . Then  $\Gamma \cong \text{Cos}(G, G_v, a)$ .*

*Conversely, let  $H$  be a core-free subgroup of a finite group  $G$  and let  $a \in G$  be such that  $G = \langle H, a \rangle$  and  $a^{-1} \in HaH$ . Then the graph  $\Gamma = \text{Cos}(G, H, a)$  is connected and  $G$ -arc-transitive. The valency of  $\Gamma$  is  $|HaH|/|H|$  and the neighbourhood  $\Gamma(H)$  of the vertex  $H \in \text{V}\Gamma$  is the set  $\Gamma(H) = \{Hah : h \in H\}$ .*

Given a connected graph  $\Gamma$ , a subgroup  $G \leq \text{Aut}(\Gamma)$  and a normal subgroup  $N \trianglelefteq G$ , the *normal quotient graph*  $\Gamma/N$  is the graph with vertex set the orbit space  $\text{V}\Gamma/N = \{v^N : v \in \text{V}\Gamma\}$  and with two orbits  $u^N$  and  $v^N$  adjacent in  $\Gamma/N$  whenever there exists a pair of vertices  $u', v' \in \text{V}\Gamma$  with  $u' \in u^N$  and  $v' \in v^N$ . Note that there exists a natural (but possibly not faithful) action of  $G/N$  on  $\Gamma/N$ . Moreover, if  $G$  is transitive on the vertices (arcs, respectively) of  $\Gamma$ , then  $G/N$  is transitive on vertices (arcs, respectively) of  $\Gamma/N$ .

A special case of normal quotients arises when the quotient projection  $\pi : \Gamma \rightarrow \Gamma/N$ ,  $v \mapsto v^N$ , is locally bijective (that is,  $\pi$  maps the neighbourhood of an arbitrary

vertex  $v \in V\Gamma$  bijectively onto the neighbourhood of  $\pi(v)$  in  $\Gamma/N$ . It is well known that, in this case,  $G/N$  acts faithfully on  $V(\Gamma/N)$ , and that the vertex-stabilisers  $G_v$  and  $(G/N)_{\pi(v)}$  are isomorphic and induce permutation isomorphic local groups  $G_v^{\Gamma(v)}$  and  $(G/N)_{\pi(v)}^{\Gamma/N(\pi(v))}$ . When the quotient projection  $\pi: \Gamma \rightarrow \Gamma/N$  is locally bijective, we will say that the pair  $(\Gamma, G)$  is an  $N$ -cover of the pair  $(\Gamma/N, G/N)$ . If, in addition,  $N$  is contained in the centre of  $G$ , then we say that the pair  $(\Gamma, G)$  is a *central  $N$ -cover* of the pair  $(\Gamma/N, G/N)$ .

We will need the following lemma describing the relationship between covers and coset graphs.

**Lemma 2.** *Let  $G$  be a group generated by a core-free subgroup  $H$  and an element  $a$ . Further, let  $\Gamma = \text{Cos}(G, H, a)$  and let  $N$  be a normal subgroup of  $G$  not containing  $a$  and intersecting the set  $H^a H$  trivially. Let  $\bar{G} = G/N$ , let  $\bar{H} = HN/N$ , and let  $\bar{a} = Na \in G/N$ . Then  $\Gamma/N \cong \text{Cos}(\bar{G}, \bar{H}, \bar{a})$  and  $(\Gamma, G)$  is an  $N$ -cover of  $(\Gamma/N, \bar{G})$ .*

*Proof.* Let  $v$  denote the vertex of  $\Gamma = \text{Cos}(G, H, a)$  corresponding to the coset  $H \in G/H$ . Then  $H = G_v$  and  $H^a = G_{v^a}$ . As  $\Gamma$  is  $G$ -arc-transitive, to show that the quotient projection  $\pi: \Gamma \rightarrow \Gamma/N$  is locally bijective, it suffices to show that the  $N$ -orbit of  $v^a$  intersects the neighbourhood of  $v$  only in  $v^a$ , that is,  $(v^a)^N \cap (v^a)^H = \{v^a\}$ . Now, if  $u \in (v^a)^N \cap (v^a)^H$ , then  $u = v^{az} = v^{ah}$  for some  $z \in N$  and  $h \in H$ . Therefore  $azh^{-1}a^{-1} \in G_v = H$ , and thus  $z \in H^a H$ . Since  $N \cap H^a H = 1$ , this implies that  $z = 1$  and that  $u = v^a = v^{ah}$ , and hence  $(v^a)^N \cap (v^a)^H = \{v^a\}$ . This shows that  $\pi: \Gamma \rightarrow \Gamma/N$  is indeed a covering projection and hence  $(\Gamma, G)$  is an  $N$ -cover of  $(\Gamma/N, G/N)$ . It follows that  $G/N$  acts faithfully and arc-transitively on  $\Gamma/N$ . The stabiliser of the vertex  $v^N$  in  $G/N$  is the group  $\bar{H} = HN/N \cong H/(H \cap N) \cong H$ , and  $\bar{a}$  maps the vertex  $\pi(v)$  to the neighbour  $\pi(v^a)$ . By Lemma 1 we may thus conclude that  $\Gamma/N \cong \text{Cos}(\bar{G}, \bar{H}, \bar{a})$ , as claimed.  $\square$

### 3. THE FAMILY OF GRAPHS WITH EXPONENTIAL GROWTH OF THE VERTEX-STABILISER

In this section we describe the family of graphs  $C(r, s)$  mentioned in Section 1, which generalise the graphs  $W_r$ . We give a definition which is slightly different, but equivalent to the definition used in [5], where they were first introduced.

Let  $C(r, 1) = W_r$ . Let  $s$  be an integer satisfying  $2 \leq s \leq r - 2$  and let  $C(r, s)$  be the graph with vertices being the  $(s - 1)$ -paths of  $C(r, 1)$  containing at most one vertex from each fibre  $V_j$ ,  $j \in \mathbb{Z}_r$ , and with two such  $(s - 1)$ -paths being adjacent in  $C(r, s)$  if and only if their intersection is an  $(s - 2)$ -path in  $C(r, 1)$ . The number of vertices of  $C(r, s)$  is clearly

$$(2) \quad |VC(r, s)| = r2^s.$$

It is easy to see that the girth of  $C(r, s)$  is 4. Further,  $C(r, s)$  is bipartite provided that  $r$  is even.

For  $i \in \mathbb{Z}_r$ , let  $x_i$  denote the automorphism of  $W_r$  which interchanges the two vertices in the fibre  $V_i$  and fixes all other vertices. Further, let  $a$  be the automorphism of  $W_r$  which maps each  $(v, i) \in VW_r$  to  $(v + 1, i)$ , and let  $b$  be the automorphism acting on the vertices of  $W_r$  according to the rule  $(v, i)^b = (-v, i)$  for every  $v \in \mathbb{Z}_r$  and  $i \in \mathbb{Z}_2$ . Then the group

$$(3) \quad G_r = \langle x_0, \dots, x_{r-1} \rangle \rtimes \langle a, b \rangle \cong C_2^r \rtimes D_r$$

acts arc-transitively on  $W_r$ . It was shown in [5] that, if  $r \neq 4$ , then  $\text{Aut}(W_r) = G_r$  ( $W_4 \cong K_{4,4}$  and hence  $\text{Aut}(W_4) \cong \text{Sym}(4) \text{ wr } \text{Sym}(2)$ ).

Since  $G_r$  permutes the  $(s-1)$ -paths of  $W_r$  containing at most one vertex from each fibre  $V_j$ , the group  $G_r$  acts as a group of automorphisms of  $C(r, s)$ . The stabiliser in  $G = G_r$  of the vertex  $v$  of  $C(r, s)$  corresponding to the  $(s-1)$ -path  $(r-s, 0)(r-s+1, 0) \dots (r-1, 0)$  in  $W_r$  is the group  $H = \langle x_0, x_1, \dots, x_{r-s-1}, b_s \rangle$ , where  $b_s$  is the element of  $G$  acting as  $(v, i)^{b_s} = (r-s-1-i, i)$ . Note that  $G = \langle H, a \rangle$ . Using Lemma 1, it is now easy to see that the graphs  $C(r, s)$  can be defined in terms of coset graphs as follows.

**Lemma 3.** *The graph  $C(r, s)$  is isomorphic to the coset graph  $\text{Cos}(G_{r,s}, H_{r,s}, a)$  where*

$$\begin{aligned} G_{r,s} &= \langle x_0, \dots, x_{r-1}, a, b \mid x_0^2 = \dots = x_{r-1}^2 = a^r = b^2 = (ab)^2 = 1, \\ &\quad x_i^a = x_{i+1}, x_i^b = x_{r-s-1-i} \rangle, \\ H_{r,s} &= \langle x_0, \dots, x_{r-1}, b \rangle \leq G_{r,s}. \end{aligned}$$

Let us finish this section by reporting the following result from [5] regarding the automorphism group of  $C(r, s)$ .

**Lemma 4.** [5, Lemma 2.12] *Let  $\Gamma = C(r, s)$  with  $1 \leq s \leq r-1$ . If  $r \neq 4$ , then  $\text{Aut}(\Gamma) = G_{r,s}$ . Moreover,  $\text{Aut}(C(4, 1)) \cong \text{Aut}(K_{4,4}) \cong \text{Sym}(4) \text{ wr } \text{Sym}(2)$ ,  $\text{Aut}(C(4, 2)) \cong \text{Sym}(2) \text{ wr } \text{Sym}(4)$  and  $\text{Aut}(C(4, 3)) \cong (\text{Sym}(2) \text{ wr } D_4) \cdot \text{Sym}(2)$ .*

#### 4. THE GRAPHS ATTAINING THE BOUND (1)

In this section, for every  $t \geq 2$ , we construct two locally- $D_4$  pairs  $(\Gamma_t^+, G_t^+)$  and  $(\Gamma_t^-, G_t^-)$  with  $|\text{V}\Gamma_t^\pm| = t2^{t+2}$  and  $|G_t^\pm| = t2^{2t+3}$ . Since  $|(G_t^\pm)_v| = |G_t^\pm|/|\text{V}\Gamma_t^\pm| = 2^{t+1}$ , we see that the pairs  $(\Gamma_t^\pm, G_t^\pm)$  indeed meet the bound (1) stated in Section 1.

Let  $t$  be an integer satisfying  $t \geq 2$ . We start by considering the *extraspecial* group  $E_t$  of order  $2^{2t+1}$  of *plus type*, which has the following presentation:

$$(4) \quad \begin{aligned} E_t = \langle x_0, \dots, x_{2t-1}, z \mid & x_i^2 = z^2 = [x_i, z] = 1 \text{ for } 0 \leq i \leq 2t-1, \\ & [x_i, x_j] = 1 \text{ for } |i-j| \neq t, \\ & [x_i, x_{t+i}] = z \text{ for } 0 \leq i \leq t-1 \rangle. \end{aligned}$$

We will now extend the group  $E_t$  by the dihedral group

$$(5) \quad D_{2t} = \langle a, b \mid a^{2t} = b^2 = 1, a^b = a^{-1} \rangle,$$

using two different 2-cocycles. In both extensions, the generators  $a$  and  $b$  will act upon the generators of  $E_t$  according to the rules:

$$x_i^a = x_{i+1} \text{ and } x_i^b = x_{t-1-i} \text{ for } 0 \leq i \leq 2t-1 \text{ (with indices taken mod } 2t).$$

To obtain the split extension  $G_t^+$ , we let  $a^{2t} = b^2 = 1$ , and thus define:

$$(6) \quad G_t^+ = E_t \rtimes D_{2t}, \quad a^{2t} = b^2 = 1, \quad x_i^a = x_{i+1}, \quad x_i^b = x_{t-1-i}.$$

The second extension  $G_t^-$  is non-split, we have  $a^{2t} = z$  and  $b^2 = 1$ :

$$(7) \quad G_t^- = E_t \cdot D_{2t}, \quad a^{2t} = z, \quad b^2 = 1, \quad x_i^a = x_{i+1}, \quad x_i^b = x_{t-1-i}.$$

Finally, let

$$(8) \quad H_t^\pm = \langle x_0, \dots, x_{t-1}, b \rangle \leq G_t^\pm,$$

and observe that  $H_t^+ \cong H_t^- \cong C_2^t \rtimes C_2$ . The graphs  $\Gamma_t^+$  and  $\Gamma_t^-$  are now defined as coset graphs on the groups  $G_t^+$  and  $G_t^-$ , respectively:

**Definition 5.** Let  $t \geq 2$  and let  $a$ ,  $H_t^\pm$  and  $G_t^\pm$  be as above. Then  $\Gamma_t^+ = \text{Cos}(G_t^+, H_t^+, a)$  and  $\Gamma_t^- = \text{Cos}(G_t^-, H_t^-, a)$ .

Before stating the main theorem of this section, let us first show that for any  $t \geq 2$ , a triple  $(G, H, a) = (G_t^\pm, H_t^\pm, a)$  satisfies the conditions of Lemma 1 and thus gives rise to a connected  $G$ -arc-transitive graph  $\Gamma_t^\pm$ . Observe first that since  $a$  cyclically permutes the elements of the generating set  $\{x_0, \dots, x_{2t-1}\}$  of  $E$ , and since  $x_0 \in H$ , the group  $\langle H, a \rangle$  contains the subgroup  $E$ . Since  $b \in H$ , we see that  $\langle H, a \rangle = \langle H, b, a \rangle = \langle E, b, a \rangle = G$ . The graph  $\Gamma_t^\pm$  is therefore connected.

To see that  $H$  is core-free in  $G$  observe that

$$H \cap H^{a^t} = \langle x_0, \dots, x_{t-1}, b \rangle \cap \langle x_t, \dots, x_{2t-1}, b \rangle = \langle b \rangle.$$

Since  $b \notin H^a$ , we see that  $H \cap H^{a^t} \cap H^a = 1$ , implying that the core of  $H$  in  $G$  is trivial. Finally, since  $b \in H$ , it follows that  $HaH = HbabH = Ha^{-1}H$ , and hence  $a^{-1} \in HaH$ . In view of Lemma 1, this implies that the graph  $\Gamma_t^\pm$  is indeed connected and  $G$ -arc-transitive.

**Theorem 6.** Let  $t$  be an integer with  $t \geq 2$  and let  $(\Gamma, G)$  be either  $(\Gamma_t^+, G_t^+)$  or  $(\Gamma_t^-, G_t^-)$ . Then the following statements hold:

- (i)  $|\text{VF}| = t2^{t+2}$  and  $|G_v| = 2^{t+1}$ , and hence the bound (1) in Section 1 is met;
- (ii)  $(\Gamma, G)$  is a central  $C_2$ -cover of the pair  $(C(2t, t), G_{2t, t})$ , where  $G_{2t, t}$  is as in Lemma 3;
- (iii)  $(\Gamma, G)$  is a locally- $D_4$  pair;
- (iv) the girth of  $\Gamma$  is 4 if  $\Gamma = \Gamma_2^+$ , is 6 if  $\Gamma = \Gamma_3^+$ , and is 8 otherwise;
- (v) if  $(\Gamma, G) \neq (\Gamma_2^-, G_2^-)$ , then  $\text{Aut}(\Gamma) = G$ , while  $|\text{Aut}(\Gamma_2^-) : G_2^-| = 9$ .

*Proof.* Let  $E = E_t$  and let  $H = H_t^+$  or  $H_t^-$ , so that  $\Gamma = \text{Cay}(G, H, a)$ . To prove part (i), note that  $|G| = |E_t| |D_{2t}| = 2^{2t+1} 4t = t2^{2t+3}$  and that  $H \cong \langle x_0, \dots, x_{t-1} \rangle \rtimes \langle b \rangle \cong C_2^t \rtimes C_2$ , implying that  $|G_v| = |H| = 2^{t+1}$ . Therefore,  $|\text{VF}| = |G|/|H| = t2^{t+2}$ , as claimed.

To prove part (ii), first observe that  $\langle z \rangle \cong C_2$  is contained in the centre of  $G$ , that  $G/\langle z \rangle \cong G_{2t, t}$  and that the natural isomorphism between  $G/\langle z \rangle$  and  $G_{2t, t}$  maps the group  $H$  bijectively onto the group  $H_{2t, t} \leq G_{2t, t}$  (defined in Lemma 3), and the element  $a \in G$  from the definition of the graph  $\Gamma = \text{Cos}(G, H, a)$  to the element  $a$  from the definition of the graph  $C(2t, t) = \text{Cos}(G_{2t, t}, H_{2t, t}, a)$ .

We now show that  $z \notin H^a H$ . Suppose, by contradiction, that  $z \in H^a H$ . Every element of  $H$  is of the form  $eb^\epsilon$  for some  $e \in \langle x_0, \dots, x_{t-1} \rangle \leq E$  and  $\epsilon \in \{0, 1\}$ , hence  $z$  can be written in the form  $(eb^\epsilon)^a (e'b^{\epsilon'})$ . Since  $E$  is normal in  $G$ , we have  $z = c(b^\epsilon)^a b^{\epsilon'}$ , for some  $c \in E$ . Since  $z \in E$ , it follows that  $(b^\epsilon)^a b^{\epsilon'} = 1$  and hence  $\epsilon = \epsilon' = 0$ . It follows that  $z$  can be written in the form  $(x_0^{\epsilon_0} \dots x_{t-1}^{\epsilon_{t-1}})^a (x_0^{\epsilon'_0} \dots x_{t-1}^{\epsilon'_{t-1}}) = (x_1^{\epsilon_0} \dots x_t^{\epsilon_{t-1}})(x_0^{\epsilon'_0} \dots x_{t-1}^{\epsilon'_{t-1}})$ . If  $\epsilon_{t-1} = 0$ , then the latter belongs to the elementary abelian group  $\langle x_0, \dots, x_{t-1} \rangle$ , which does not contain  $z$ . Similarly, if  $\epsilon'_0 = 0$ , then the latter belongs to the elementary abelian group  $\langle x_1, \dots, x_t \rangle$ , which does not contain  $z$ . Hence we may assume that  $\epsilon_{t-1} = \epsilon'_0 = 1$ . Now, since  $x_t x_0 = x_0 x_t z$ , it follows

that

$$\begin{aligned}
z &= x_1^{\epsilon_0} \cdots x_{t-1}^{\epsilon_{t-2}} x_t x_0 x_1^{\epsilon'_1} \cdots x_{t-1}^{\epsilon'_{t-1}} \\
&= x_1^{\epsilon_0} \cdots x_{t-1}^{\epsilon_{t-2}} x_0 x_t z x_1^{\epsilon'_1} \cdots x_{t-1}^{\epsilon'_{t-1}} \\
&= x_1^{\epsilon_0} \cdots x_{t-1}^{\epsilon_{t-2}} x_0 x_1^{\epsilon'_1} \cdots x_{t-1}^{\epsilon'_{t-1}} x_t z = dx_t z
\end{aligned}$$

with  $d \in \langle x_0, \dots, x_{t-1} \rangle$ . Therefore  $x_t \in \langle x_0, \dots, x_{t-1} \rangle$ , which is clearly a contradiction. Thus we have shown that  $z \notin H^a H$ . It follows by Lemma 2 that  $\Gamma/\langle z \rangle \cong C(2t, t)$  and that  $\Gamma$  is a  $\langle z \rangle$ -cover of  $C(2t, t)$ . Part (ii) of the theorem is thus proved.

Moreover, since the pair  $(C(2t, t), G_{2t, t})$  is locally- $D_4$ , so is the covering pair  $(\Gamma, G)$ , thus proving part (iii).

In the proof of parts (iv) and (v) we will need detailed information about the spheres of radius 2 and 3 around the vertex  $H$ . For  $i \geq 1$  and  $v \in V\Gamma$ , let  $\Gamma_i(v)$  denote the set of vertices in  $\Gamma$  at distance  $i$  from  $v$ . To determine the neighbourhood  $\Gamma_1(H)$ , observe that  $Ha, Ha^{-1} = Hab, Hx_{2t-1}a = Hax_0$  and  $Hx_t a^{-1} = Ha^{-1}x_{t-1} = Habx_{t-1}$  are four pairwise distinct cosets of the form  $Hah$  with  $h \in H$ . Since  $\Gamma$  has valency 4, this implies that

$$(9) \quad \Gamma_1(H) = \{Hg : g \in X_1\}, \text{ where } X_1 = \{a, x_{2t-1}a, a^{-1}, x_t a^{-1}\}.$$

Further, observe that

$$(10) \quad \Gamma_2(H) = \{Hg : g \in X_2\} \setminus \{H\}, \text{ where } X_2 = \{gh : g, h \in X_1\}.$$

An easy computation shows that

$$\begin{aligned}
(11) \quad X_2 &= \{x_{2t-2}^{\epsilon_1} x_{2t-1}^{\epsilon_2} a^2 : \epsilon_1, \epsilon_2 \in \{0, 1\}\} \cup \\
&\quad \{x_t^{\epsilon_1} x_{t+1}^{\epsilon_2} a^{-2} : \epsilon_1, \epsilon_2 \in \{0, 1\}\} \cup \\
&\quad \{x_t, x_t z, x_{2t-1}, x_{2t-1} z\}.
\end{aligned}$$

Similarly, note that

$$(12) \quad \Gamma_3(H) = \{Hg : g \in X_3\} \setminus \Gamma_1(H), \text{ where } X_3 = \{gh : g, h \in X_1, X_2\}.$$

By a straightforward computation we get

$$\begin{aligned}
(13) \quad X_3 &= \{x_{2t-3}^{\epsilon_1} x_{2t-2}^{\epsilon_2} x_{2t-1}^{\epsilon_3} a^3 : \epsilon_1, \epsilon_2, \epsilon_3 \in \{0, 1\}\} \cup \\
&\quad \{x_t^{\epsilon_1} x_{t+1}^{\epsilon_2} x_{t+2}^{\epsilon_3} a^{-3} : \epsilon_1, \epsilon_2, \epsilon_3 \in \{0, 1\}\} \cup \\
&\quad \{x_t^{\epsilon_1} x_{2t-1}^{\epsilon_2} z^{\epsilon_3} a : \epsilon_1, \epsilon_2, \epsilon_3 \in \{0, 1\}\} \cup \\
&\quad \{x_{2t-2} x_{2t-1}^{\epsilon_1} z^{\epsilon_2} a : \epsilon_1, \epsilon_2 \in \{0, 1\}\} \cup \\
&\quad \{x_t^{\epsilon_1} x_{t+1}^{\epsilon_2} z^{\epsilon_3} a^{-1} : \epsilon_1, \epsilon_2, \epsilon_3 \in \{0, 1\}\} \cup \\
&\quad \{x_t^{\epsilon_1} x_{2t-1} z^{\epsilon_2} a : \epsilon_1, \epsilon_2 \in \{0, 1\}\}.
\end{aligned}$$

Having computed the second and the third neighbourhood of the vertex  $H$ , it is now easy to determine the girth of the graph  $\Gamma$ . Recall first that  $\Gamma$  is a 2-fold cover of the graph  $C(2t, t)$ , which is bipartite and of girth 4. This implies that  $\Gamma$  is also bipartite and of girth not exceeding 8.

Now, if  $t = 2$  and  $\Gamma = \Gamma_t^+$ , then the order of  $a$  is 4, and hence the elements  $a^{-2}, a^2$ , listed in (11) coincide. In particular,  $\Gamma$  contains the 4-cycle  $(H, Ha, Ha^2, Ha^{-1})$ , and thus the girth of  $\Gamma$  is 4.

In all the other cases (that is, if  $\Gamma = \Gamma_2^-$  or if  $t \geq 3$ ), the 12 elements of  $X_2$  listed in (11) are a transversal of 12 pairwise distinct  $H$ -cosets. This implies that the girth of  $\Gamma$  is at least 6. If  $t = 3$  and  $\Gamma = \Gamma_t^+$ , then  $a^3 = a^{-3}$  and therefore  $\Gamma$  contains the 6-cycle  $(H, Ha, H^2, Ha^3, Ha^{-2}, Ha^{-1})$ . In particular, the girth of  $\Gamma_3^+$  is 6.

In all the other cases (that is, if  $\Gamma = \Gamma_3^-$  or if  $t \geq 4$ ), the 36 elements of  $X_3$ , listed in (13), are pairwise distinct, and in fact are a transversal of 36 pairwise distinct  $H$ -cosets. The girth of  $\Gamma$  is thus at least (and therefore exactly) 8. This proves part (iv) of the theorem.

Let us now prove part (v). The automorphism groups of  $\Gamma_t^\pm$  for  $t \leq 3$  can be checked easily with Magma [1]. We will therefore assume that  $t \geq 4$  and set  $\Gamma = \Gamma_t^\pm$  and  $A = \text{Aut}(\Gamma)$ .

We will first show that the orbits of  $\langle z \rangle$  on  $V\Gamma$  form a system of imprimitivity for the action of  $A$  on  $V\Gamma$ . Set

$$(14) \quad \mathcal{C} = \bigcap_{u \in \Gamma(H)} \Gamma_3(u) = \Gamma_3(H)a \cap \Gamma_3(H)x_{2t-1}a \cap \Gamma_3(H)a^{-1} \cap \Gamma_3(H)x_t a^{-1}$$

and set  $\mathcal{B} = \mathcal{C} \cup \{H\}$ . Using (13), a straightforward calculation shows that

$$(15) \quad \mathcal{B} = \{H, Hx_t x_{2t-1}, Hz, Hx_t x_{2t-1}z\}.$$

We claim that  $\mathcal{B}$  is a block of imprimitivity for  $A$ . Observe first that the setwise stabiliser  $A_{\mathcal{B}}$  of the set  $\mathcal{B}$  in  $A$  acts transitively on  $\mathcal{B}$  and contains the vertex stabiliser  $A_H$ . The latter follows directly from the definition of the set  $\mathcal{B}$ , while the former follows from the observation that the group  $\langle x_t x_{2t-1}, z \rangle$  preserves  $\mathcal{B}$  and acts transitively upon it. In particular,  $A_{\mathcal{B}} = A_H \langle x_t x_{2t-1}, z \rangle$ . Hence  $\mathcal{B}$  is an orbit of a subgroup of  $A$  which strictly contains the vertex-stabiliser  $A_H$ . This shows that  $\mathcal{B}$  is a block of imprimitivity for  $A$ .

Now observe that each of the vertices  $H$  and  $Hx_t x_{2t-1}$  has a neighbour in each of the four translates  $\mathcal{B}a, \mathcal{B}a^{-1}, \mathcal{B}x_{2t-1}a, \mathcal{B}x_t a^{-1}$  of the block  $\mathcal{B}$  (this is obviously true for  $H$ , and follows easily for  $Hx_t x_{2t-1}$  from the fact that  $z$  is contained in the centre of  $G$ ). On the other hand, a direct inspection shows that  $Hx_t x_{2t-1}$  and  $Hx_t x_{2t-1}z$  have no neighbours in these four translates of  $\mathcal{B}$ . In particular, the stabiliser  $A_H$  cannot map the vertex  $Hx_t x_{2t-1}$  to any of the other two vertices  $Hx_t x_{2t-1}$  and  $Hx_t x_{2t-1}z$  in the block  $\mathcal{B}$ . In particular,  $A_H$  fixes the vertex  $Hx_t x_{2t-1}$ , and hence  $A_H = A_{Hx_t x_{2t-1}}$ . It follows that the group  $\langle A_H, z \rangle$  preserves the set  $\{H, Hx_t x_{2t-1}\}$ , acts upon it transitively, and contains the vertex-stabiliser  $A_H$ . This implies that its orbit  $\{H, Hx_t x_{2t-1}\}$  is a block of imprimitivity for  $A$ , as claimed.

Now consider the kernel  $K$  of the action of  $A$  on the  $\langle z \rangle$ -orbits. Since  $\Gamma \rightarrow \Gamma/\langle z \rangle$  is a covering projection, we know that  $K$  acts semiregularly on  $V\Gamma$ . On the other hand  $\langle z \rangle \leq K$  has the same orbits on  $V\Gamma$  as  $K$ , and hence  $K = K_v \langle z \rangle = \langle z \rangle$ . In particular,  $\langle z \rangle$  is normal in  $A$ . Therefore  $A/\langle z \rangle \leq \text{Aut}(\Gamma/\langle z \rangle) \cong \text{Aut}(C(2t, t)) \cong G_{2t, t}$  (see Lemma 4). In particular,  $|A| = 2|G_{2t, t}| = |G|$ , and therefore  $A = G$ .  $\square$

**Remark 7.** Since  $G_t^+$  and  $G_t^-$  are non-isomorphic groups, it follows from Theorem 6 that  $\text{Aut}(\Gamma_t^+) \not\cong \text{Aut}(\Gamma_t^-)$ . In particular, we obtain that  $\Gamma_t^+$  and  $\Gamma_t^-$  are non-isomorphic graphs. Moreover, since the girth of the graphs  $C(r, s)$  is 4, none of the graphs  $\Gamma_t^\pm$ , other than possibly  $\Gamma_2^+$ , is isomorphic to any of the graphs  $C(r, s)$ . On the other hand, it can be easily checked that  $\Gamma_2^+ \cong C(4, 3)$ .

### 5. A FAMILY OF $\text{Sym}(n)$ -ARC-TRANSITIVE GRAPHS

In this section, we introduce another interesting family of tetravalent  $G$ -arc-transitive graphs with arbitrarily large vertex-stabilisers. Unlike the graphs  $\Gamma_t^\pm$ , which have soluble automorphism groups, the graphs we are going to describe here have an almost simple arc-transitive group of automorphisms.

Let  $m \geq 2$  be an integer and let  $G$  be the symmetric group  $\text{Sym}(4m)$  acting on the set  $\{1, 2, \dots, 4m\}$ . Define the following permutations of  $G$

$$\begin{aligned} x_i &= (2i-1, 2i) && \text{for } 1 \leq i \leq 2m-1, \\ h &= (4m-1, 4m) \prod_{i=1}^{m-1} (2i-1, 4m-2i-1)(2i, 4m-2i) \\ a &= (4m-2, 4m) \prod_{i=1}^{m-1} (2i-1, 4m-2i-3)(2i, 4m-2i-2) \\ g &= (1, 3, 5, \dots, 4m-3)(2, 4, 6, \dots, 4m-4, 4m-2, 4m-1, 4m). \end{aligned}$$

We can now define the graphs  $\Delta_m$ .

**Definition 8.** For  $G = \text{Sym}(4m)$  and  $H = \langle x_1, x_2, \dots, x_{2m-1}, h \rangle$ , let  $\Delta_m = \text{Cos}(G, H, a)$ .

Before proving that the graphs  $\Delta_m$  are indeed connected tetravalent graphs, we first observe that the following holds:

$$(16) \quad \begin{aligned} g &= ah, \\ x_i^h &= x_{2m-i} && \text{for } 1 \leq i \leq 2m-1, \\ x_i^g &= x_{i+1} && \text{for } 1 \leq i \leq 2m-2, \\ x_{2m-1}^g &= (4m-3, 4m-2)^g = (1, 4m-1). \end{aligned}$$

In particular, the group  $H$  is isomorphic to a semidirect product  $C_2^{2m-1} \rtimes C_2$  and has order  $2^{2m}$ .

**Theorem 9.** For any  $m \geq 2$ , the graph  $\Delta_m$  is non-bipartite, connected, tetravalent and  $G$ -arc-transitive. Moreover,  $|\text{V}\Delta_m| = (4m)!/2^{2m}$  and  $|G_v| = 2^{2m}$ .

*Proof.* Let first prove that the triple  $(G, H, a)$  from Definition 8 satisfies the conditions stated in Lemma 1. In other words, let us prove that the core of  $H$  in  $G$  is trivial, that  $G = \langle H, a \rangle$ , and that  $Ha^{-1}H = HaH$ . Since  $a$  is an involution, the latter condition is automatically fulfilled. Furthermore, since the only non-trivial proper normal subgroup of  $G = \text{Sym}(4m)$  is the group  $\text{Alt}(4m)$  and since  $H \neq \text{Alt}(4m)$ , it follows that the core of  $H$  in  $G$  is trivial. It remains to see that  $G$  is generated by  $H$  and  $a$ .

Set  $K = \langle H, a \rangle$  and observe that from Equation (16),  $g \in K$  and hence  $G$  is a transitive subgroup of  $\text{Sym}(4m)$ . Furthermore,  $g$  is a product of two cycles of lengths  $2m-1$  and  $2m+1$ . Since  $2m-1$  and  $2m+1$  are coprime, it follows that  $K$  is a primitive subgroup of  $\text{Sym}(4m)$ . As  $K$  contains the transposition  $x_1 = (1, 2)$ , we obtain from [3, Theorem 3.3A(ii)] that  $K = \text{Sym}(4m) = G$ . Lemma 1 now implies that  $\Delta_m$  is a connected  $G$ -arc-transitive graph with  $G_v \cong H$ .

Finally, recall that  $x_i^a = x_{2m-i-1}$  for  $i = 1, \dots, 2m-2$ . In particular,  $H \cap H^a \geq \langle x_1, \dots, x_{2m-2} \rangle$  and  $|H : H \cap H^a|$  divides 4. Since  $H \cap H^a$  is the stabiliser in  $G$  of the arc  $(H, Ha)$  and since  $\Delta_m$  is  $G$ -arc-transitive, we obtain that the valency of  $\Delta_m$  divides 4. As  $G$  is almost simple and acts faithfully on  $\Delta_m$ , we have that  $\Delta_m$



has valency 4. Since  $H \not\leq \text{Alt}(4m)$ , the group  $\text{Alt}(4m)$  is transitive on  $V\Delta_m$ . As  $\text{Alt}(4m)$  is the only subgroup of index 2 in  $G$ , we see that  $\Delta_m$  is non-bipartite.  $\square$

Let  $\Gamma = \Delta_m$  as in Definition 8 and let  $G = \text{Sym}(4m) \leq \text{Aut}(\Gamma)$ . We consider the growth rate of the quantity  $x = |G_v| = 2^{2m}$  with respect to  $y = |\text{VT}| = (4m)!/2^{2m}$ . Using Stirling's formula for the factorial term in  $y$ , one easily obtains that

$$y \approx \sqrt{4\pi \log(x)} x^{1-2 \log(e)} \log(x)^{\log(x^2)},$$

with all the logarithms having base 2. Since  $\log(x)^{\log(x^2)} = x^{2 \log(x) \log(\log(x))}$ , this shows that  $x = |G_v|$  grows faster than any logarithmic function on  $y = |\text{VT}|$  but slower than  $|\text{VT}|^c$  for any  $c > 0$ . The graphs  $\Delta_m$  are thus quite far from attaining the bound given in (1), yet we conjecture that a much faster growth of  $|G_v|$  cannot be expected when  $G$  is almost simple.

**Conjecture 10.** *For any positive constant  $c$  there exists a finite family of graphs  $\mathcal{F}_c$ , such that the following holds: if  $G$  is an almost simple group and  $\Gamma$  is a connected tetravalent  $G$ -arc-transitive graph such that  $\Gamma$  is not contained in  $\mathcal{F}_c$ , then  $|G_v| < |\text{VT}|^c$ .*

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